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Coarse grained and fine dynamics in trapped ion Raman schemes

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Abstract
A novel result concerning Raman coupling schemes implemented using trapped ions is obtained. By means of an operator perturbative approach, it is shown that the complete time evolution of these systems can be expressed, with a high degree of accuracy, as the product of two unitary evolutions. The first one describes the time evolution related to an effective coarse grained dynamic. The second is a suitable correction restoring the *fine* dynamics suppressed by the coarse graining performed to adiabatically eliminate the nonresonantly coupled atomic level. The case where a decoherence source is present is also studied.

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1. Introduction

Trapped ions provide an effective platform for observing interesting aspects of quantum mechanics and for realizing useful applications in the context of quantum computation [1–3].

In these traps, a time-dependent quadrupolar electromagnetic field is responsible for a charged particle motion which may be kinematically assimilated to the motion of a massive spot subjected to a quadratic potential. Such a circumstance provides the possibility of describing the centre of mass of an ion confined into a rf Paul trap as a quantum harmonic oscillator [4–6]. In addition, the ion possesses atomic degrees of freedom related to the electronic motion around the nucleus [2, 4–7].

Acting upon the system via laser fields, it is possible to induce vibronic transitions described by Jaynes–Cummings-like Hamiltonians [7, 8]. Such interactions are characterized by nonlinearities governed by the so called Lamb–Dicke parameter, which in a spherically

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symmetric trap—the case we consider in this paper—is nothing but the ratio between the width of the ion vibrational ground wavefunction and the laser wavelength. Controlling the Lamb–Dicke parameter leaving unchanged the laser frequency would then provide the possibility of implementing a wider variety of Hamiltonian models. Unfortunately, these two parameters are strictly related, the Lamb–Dicke parameter being proportional to the inverse of the wavelength, hence proportional to the laser frequency. Therefore, in a spherically symmetric trap they cannot be independently adjusted.[3] Nevertheless, such a possibility can be achieved exploiting a fundamental property of Raman coupling schemes. In these couplings, a three-level system is subjected to a far detuned Λ-scheme. Under some assumptions and within approximations concerning the time scale used to observe the system, the dynamics may be well enough described via an effective Hamiltonian thinkable as a Jaynes–Cummings-like Hamiltonian related to an *effective* laser field, with frequency and wave vector given by the differences between the corresponding parameters of the two *real* Λ-scheme lasers. Then, changing the angle between the two real laser propagation directions leads to the possibility of obtaining effective lasers such that the product of their wavelength and frequency is not the velocity of light. The price to pay is the complete ignorance about the detailed system dynamics at a *fine* time scale.

In this paper, we will try to overcome such a limit by means of a new approach. More precisely, we will analyse the dynamics of a three-level trapped ion subjected to a Λ Raman coupling scheme using a very convenient perturbative decomposition of the evolution operator of the system. We will show that the Raman scheme time evolution can be factorized, at the second perturbative order (hence with a high degree of accuracy), into two unitary evolutions. The first one can be interpreted as the *effective* time evolution which may be obtained adiabatically eliminating the far detuned atomic level from the coupling scheme [9]. Such a dynamic concerns the *coarse grained* variables and is in accordance with already well-known results [9]. The second unitary evolution introduces the correction necessary to take into account the *fine* deviation from the coarse grained time evolution.

The same tools developed for studying the coherent dynamics of the Raman scheme become fruitful for investigating the case where a decoherence source is considered. In more detail, assuming that the far detuned level is an excited level with non-negligible decay rates towards the other two levels, we are able to cast the master equation of the system in a very convenient canonically equivalent form. This procedure sheds new light on the role played by the adiabatically eliminated level in the occurrence of decoherence phenomena affecting the effective coherent cycle involving the other two levels.

2. The physical system

The physical system on which we focus is a three-level harmonically trapped ion subjected to a Raman coupling scheme involving atomic transitions. The relevant Schrödinger picture

[3] In a *linear trap*—i.e. in a trap with strong confinement induced along a couple of axes and weaker harmonic binding along the third axis, the 'principal axis'—one can modify the incidence angle of the laser beam with respect to the principal axis in order to control the Lamb–Dicke parameter, which, in this case (the degrees of freedom associated with the non-principal axes being ignored), is proportional to the *projection* of the wave vector along this axis. In an isotropic trap there is no principal axis and this trick makes no sense. A trick applicable to an isotropic trap is to prepare a vibrational mode, say the one along the x-direction, for instance in a coherent state, while the other two modes are left in the ground state. Then, exciting the system on the first red vibrational sideband, it turns out that only the vibration along the x-axis contributes to the atom–light coupling, so allowing a tuning of the effective Lamb–Dicke parameter by changing the incidence angle of the laser beam. However, we stress that this trick relies on the preparation of the system in a *specific* initial state.

Hamiltonian of the Raman Λ-scheme is then given by

$$\hat{H}_\Lambda(t) = \sum_{l=1,2,3} \hbar\omega_l \hat{\sigma}_{ll} + \hbar\nu \sum_{\alpha=x,y,z} \hat{a}_\alpha^\dagger \hat{a}_\alpha + [\hbar g_{13}\, \mathrm{e}^{-\mathrm{i}(\vec{k}_{13}\cdot\vec{r}-\omega_{13}t)} \hat{\sigma}_{13} + \mathrm{h.c.}]$$
$$+ [\hbar g_{23}\, \mathrm{e}^{-\mathrm{i}(\vec{k}_{23}\cdot\vec{r}-\omega_{23}t)} \hat{\sigma}_{23} + \mathrm{h.c.}], \tag{1}$$

where $\hat{\sigma}_{lm} \equiv |l\rangle\langle m|$ (with $l, m = 1, 2, 3$), $\{|l\rangle\}$ is the considered three atomic levels and $\{\hbar\omega_l\}$ are the corresponding energies; $\hat{a}_\alpha (\alpha = x, y, z)$ is the annihilation operator related to the centre of mass harmonic motion along the direction α (we will denote the associated Fock basis by $\{\psi_n^\alpha\}$):

$$\hat{a}_x = \left(\frac{\mu\nu}{2\hbar}\right)^{1/2} \left(\hat{x} + \frac{\mathrm{i}}{\mu\nu}\hat{p}_x\right), \ldots$$

with μ denoting the mass of the ion. For the sake of simplicity (but without loss of generality), we have assumed to deal with a 3D degenerate parabolic trap with single frequency ν. The two laser fields responsible for the coupling terms are characterized by complex strengths (proportional to the laser amplitude and to the atomic dipole operator, and including the laser phases), wave vectors and frequencies $g_{13}, \vec{k}_{13}, \omega_{13}$ and $g_{23}, \vec{k}_{23}, \omega_{23}$, respectively.

The *auxiliary level* $|3\rangle$ is assumed to be dipole coupled to both the levels $|1\rangle$ and $|2\rangle$ via far detuned lasers. Precisely, the two laser frequencies are chosen in such a way that

$$\Delta \equiv \omega_3 - \omega_1 - \omega_{13} = \omega_3 - \omega_2 - \omega_{23}, \tag{2}$$

where the detuning Δ satisfies the condition

$$|\Delta| \gg |g_{13}|, |g_{23}|, \nu. \tag{3}$$

The analysis of such a Hamiltonian model has been already carried out, for instance in [9], by means of the adiabatic elimination of the nonresonantly coupled atomic level $|3\rangle$, following the path pointed out in [10, 11] and including the motional degrees of freedom. Indeed, due to the large detuning, the transitions coupling levels $|1\rangle$ and $|2\rangle$ with the auxiliary level $|3\rangle$ are very fast. Therefore, considering only coarse grained observables, meaning that the system is observed at a 'rough enough time scale', effectively eliminates the far detuned level; namely, at such a time scale, the only observables, and hence meaningful dynamical behaviours, are the one involving levels $|1\rangle$ and $|2\rangle$ as a result of a time-averaging procedure which takes into account the compound processes having $|3\rangle$ as an intermediate virtual level. Anyhow, this procedure suppresses the *fine* dynamics, that is it sacrifices any information concerning the fast dynamics the auxiliary level is involved in.

It must be also stressed that if the auxiliary level is an excited level with non-negligible decay rates towards levels $|1\rangle$ and $|2\rangle$ the observable effects of the fast oscillations involving level $|3\rangle$ should not be deleted by the coarse graining. In fact, one should expect that the transitions to the auxiliary level, composed with decays, can gradually introduce decoherence into the effective coherent cycle involving levels $|1\rangle$ and $|2\rangle$. This conjecture is actually compatible with experimental observations [12]. We will discuss this aspect in section 5.

In the following we present a perturbative approach to the solution of the dynamical problem related to $\hat{H}_\Lambda$ overcoming the limit of the coarse graining, making it possible to study also the *fine* dynamics discarded by the adiabatic elimination. The first step consists in passing to a rotating frame, meaning that the time-dependent Hamiltonian $\hat{H}_\Lambda$ is canonically transformed via the operator

$$\hat{R}(t) = \mathrm{e}^{-\mathrm{i}\hat{A}t}, \tag{4}$$

where

$$\hat{A} = (\omega_3 - \Delta)(\hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33}) - \omega_{13}\hat{\sigma}_{11} - \omega_{23}\hat{\sigma}_{22}$$
$$= \omega_1\hat{\sigma}_{11} + \omega_2\hat{\sigma}_{22} + (\omega_3 - \Delta)\hat{\sigma}_{33}, \quad (5)$$

into the following time-independent rotating frame Hamiltonian:

$$\hat{\mathfrak{H}} := \hat{R}(t)^\dagger(\hat{H}_\Lambda(t) - \hat{A})\hat{R}(t) = \hbar\Delta\hat{H} = \hbar\Delta(\hat{H}_0 + \hat{H}_B + \hat{H}_\updownarrow), \quad (6)$$

where $\hat{H}$ is a *dimensionless Hamiltonian* which is the sum of the three Hermitian operators $\hat{H}_0$, $\hat{H}_B$ and $\hat{H}_\updownarrow$ defined as

$$\begin{cases} \hat{H}_0 := \hat{\sigma}_{33}, \\ \hat{H}_B := \frac{\nu}{\Delta}\sum_{\alpha=x,y,z}\hat{a}^\dagger_\alpha\hat{a}_\alpha, \\ \hat{H}_\updownarrow := \left[\frac{g_{13}}{\Delta}\,\mathrm{e}^{-\mathrm{i}\vec{k}_{13}\cdot\vec{r}}\hat{\sigma}_{13} + \mathrm{h.c.}\right] + \left[\frac{g_{23}}{\Delta}\,\mathrm{e}^{-\mathrm{i}\vec{k}_{23}\cdot\vec{r}}\hat{\sigma}_{23} + \mathrm{h.c.}\right]. \end{cases} \quad (7)$$

Considering the assumption given by inequality (3), both $\hat{H}_B$ and $\hat{H}_\updownarrow$ may be thought of as perturbations with respect to $\hat{H}_0$. In fact, introducing the dimensionless perturbative parameter

$$\lambda := \frac{g}{\Delta}, \qquad g \equiv \max\{\nu, |g_{13}|, |g_{23}|\}, \quad (8)$$

both $\hat{H}_B$ and $\hat{H}_\updownarrow$ are first-order perturbations in λ:

$$\hat{H} = \hat{H}(\lambda) = \hat{H}_0 + \lambda\varkappa\sum_{\alpha=x,y,z}\hat{a}^\dagger_\alpha\hat{a}_\alpha + \lambda\sum_{j=1,2}[\varkappa_{j3}\,\mathrm{e}^{-\mathrm{i}\vec{k}_{j3}\cdot\vec{r}}\hat{\sigma}_{j3} + \mathrm{h.c.}], \quad (9)$$

where $\varkappa \equiv \nu/g \leqslant 1, \varkappa_{j,3} \equiv g_{j,3}/g, |\varkappa_{j,3}| \leqslant 1$, and we note that, due to condition (3), $\lambda \ll 1$. It is worth noting that the circumstance that $\hat{H}_B$ is treated as a perturbation leads to the *eccentric* situation of an unperturbed Hamiltonian, $\hat{H}_0$, wherein the bosonic degrees of freedom are *absent*. Nevertheless, as we shall see, such a mathematical artifice reveals fruitful in order to succeed in factorizing the coarse grained dynamics and its fine correction.

Our solving procedure relies on a suitable canonical transformation $\mathrm{e}^{\mathrm{i}\hat{Z}(\lambda)}$ of the rotating frame Hamiltonian such that

$$\mathrm{e}^{\mathrm{i}\hat{Z}(\lambda)}\hat{\mathfrak{H}}\,\mathrm{e}^{-\mathrm{i}\hat{Z}(\lambda)} = \hbar\Delta\,\mathrm{e}^{\mathrm{i}\hat{Z}(\lambda)}\hat{H}(\lambda)\,\mathrm{e}^{-\mathrm{i}\hat{Z}(\lambda)} = \hbar\Delta(\hat{H}_0 + \hat{C}(\lambda)), \quad (10)$$

where $\hat{C}(\lambda)$, $\hat{Z}(\lambda)$ depend analytically on the perturbative parameter λ and $\hat{C}(\lambda)$ is a *constant of motion* with respect to the unperturbed dynamics, i.e., $[\hat{H}_0, \hat{C}(\lambda)] = 0$. This transformation allows us to give a very convenient decomposition of the evolution operator associated with the rotating frame Hamiltonian, namely

$$\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{\mathfrak{H}}t\right) = \mathrm{e}^{-\mathrm{i}\hat{Z}(\lambda)}\exp(\mathrm{i}\Delta\hat{H}_0t)\exp(\mathrm{i}\Delta\hat{C}(\lambda)t)\,\mathrm{e}^{\mathrm{i}\hat{Z}(\lambda)}. \quad (11)$$

At this point, truncating the power expansions

$$\hat{C}(\lambda) = \lambda\hat{C}_1 + \lambda^2\hat{C}_2 + \cdots + \lambda^n\hat{C}_n + \cdots, \qquad \hat{Z}(\lambda) = \lambda\hat{Z}_1 + \lambda^2\hat{Z}_2 + \cdots + \lambda^n\hat{Z}_n + \cdots$$

at a given perturbative order, one obtains useful expressions of the evolution operator by formula (11). This procedure has been developed in a general setting in [13, 14]. In the next section, we want to recall briefly the mathematical background and to show how the operators $\hat{C}_1, \hat{C}_2, \ldots, \hat{Z}_1, \hat{Z}_2, \ldots$ can be computed by a suitable iterative process. Then, we will give the explicit solutions for our case up to the second perturbative order.

3. Perturbative analysis of the rotating frame Hamiltonian

Let $\hat{H}_{\rm u}$, $\hat{H}_{\rm p}$ be Hermitian operators and assume that $\hat{H}_{\rm u}$ has a purely discrete spectrum. Denote by $E_0 < E_1 < E_2 < \cdots$ the (possibly degenerate) eigenvalues of $\hat{H}_{\rm u}$ and by $\hat{P}_0, \hat{P}_1, \hat{P}_2, \ldots$ the associated eigenprojectors. Now, consider the operator $\hat{H}(\lambda) = \hat{H}_{\rm u} + \lambda\hat{H}_{\rm p}$, $\lambda \in \mathbb{C}$, which is Hermitian if λ is real. It is possible to show that, under certain conditions [15], there exist positive constants $r_0, r_1, r_2, \ldots$ and a simply connected neighbourhood $\mathcal{I}$ of zero in $\mathbb{C}$ such that the following contour integral on the complex plane

$$\hat{P}_m(\lambda) = \frac{\rm i}{2\pi}\oint_{|E-E_m|=r_m} {\rm d}E\ (\hat{H}(\lambda) - E)^{-1}, \lambda \in \mathcal{I},$$

defines a projection ($\hat{P}_m(\lambda)^2 = \hat{P}_m(\lambda)$), which is an orthogonal projection for real λ, with $\hat{P}_m(0) = \hat{P}_m$, and $\mathcal{I} \ni \lambda \mapsto \hat{P}_m(\lambda)$ is an analytic operator-valued function. Moreover, the range of $\hat{P}_m(\lambda)$ is an invariant subspace for $\hat{H}(\lambda)$, hence

$$\hat{H}(\lambda)\hat{P}_m(\lambda) = \hat{P}_m(\lambda)\hat{H}(\lambda)\hat{P}_m(\lambda), \tag{12}$$

and there exists an analytic family $\hat{U}(\lambda)$ of invertible operators such that

$$\hat{P}_m = \hat{U}(\lambda)\hat{P}_m(\lambda)\hat{U}(\lambda)^{-1}, \qquad \hat{U}(0) = {\rm Id}, \tag{13}$$

and $\hat{U}(\lambda) = {\rm e}^{{\rm i}\hat{Z}(\lambda)}$, $\lambda \in \mathcal{I}$, with $\hat{Z}(\lambda^*) = \hat{Z}(\lambda)^\dagger$ (hence, for real λ, $\hat{Z}(\lambda)$ is Hermitian and $\hat{U}(\lambda)$ is unitary), where $\mathcal{I} \ni \lambda \mapsto \hat{Z}(\lambda)$ is analytic. One can easily show that the function $\lambda \mapsto \hat{U}(\lambda)$ is not defined uniquely by condition (13) even in the simplest case when $\hat{H}_{\rm u}$ has a non-degenerate spectrum. Anyway, the nonuniqueness in the definition of $\hat{U}(\lambda)$ is not relevant if one is only interested in obtaining an expression of the evolution operator associated with $\hat{H}(\lambda)$ of the general form (11). We will see soon that there is a natural condition which fixes a unique solution for $\hat{U}(\lambda)$.

Now, let us define the operator

$$\hat{K}(\lambda) := \hat{U}(\lambda)\hat{H}(\lambda)\hat{U}(\lambda)^{-1}, \tag{14}$$

which, for real λ, is unitarily equivalent to $\hat{H}(\lambda)$. Using relations (12) and (13), we find

$$\hat{K}(\lambda)\hat{P}_m = \hat{U}(\lambda)\hat{H}(\lambda)\hat{P}_m(\lambda)\hat{U}(\lambda)^{-1} = \hat{U}(\lambda)\hat{P}_m(\lambda)\hat{H}(\lambda)\hat{P}_m(\lambda)\hat{U}(\lambda)^{-1}$$

and hence $\hat{K}(\lambda)\hat{P}_m = \hat{P}_m\hat{K}(\lambda)\hat{P}_m$. It follows that $[\hat{H}_{\rm u}, \hat{K}(\lambda)] = 0$ and then we obtain the following important decomposition formula:

$$\hat{U}(\lambda)\hat{H}(\lambda)\hat{U}(\lambda)^{-1} = \hat{H}_{\rm u} + \hat{C}(\lambda), \tag{15}$$

where $[\hat{C}(\lambda), \hat{H}_{\rm u}] = 0$, i.e., $\hat{C}(\lambda)$ is a constant of the motion with respect to the time evolution generated by $\hat{H}_{\rm u}$. At this point, we can obtain perturbative expressions of the unknown operators $\hat{C}(\lambda)$, $\hat{U}(\lambda)$ by means of a recursive algebraic procedure.

Indeed, since the functions $\lambda \mapsto \hat{C}(\lambda)$ and $\lambda \mapsto \hat{Z}(\lambda)$ are analytic in $\mathcal{I}$ and $\hat{C}(0) = \hat{Z}(0) = 0$, we can write

$$\hat{C}(\lambda) = \sum_{n=1}^{\infty}\lambda^n\hat{C}_n, \qquad \hat{Z}(\lambda) = \sum_{n=1}^{\infty}\lambda^n\hat{Z}_n, \qquad \lambda \in \mathcal{I}. \tag{16}$$

In order to determine the operators $\{\hat{C}_n\}$ and $\{\hat{Z}_n\}$, we substitute the exponential form ${\rm e}^{{\rm i}\hat{Z}(\lambda)}$ of $\hat{U}(\lambda)$ in formula (15) thus getting

$$\hat{H}(\lambda) + \sum_{n=1}^{\infty}\frac{{\rm i}^n}{n!}{\rm ad}^n_{\hat{Z}(\lambda)}\hat{H}(\lambda) = \hat{H}_{\rm u} + \hat{C}(\lambda),$$

where we recall that ${\rm ad}_{\hat{Z}(\lambda)}\hat{H}(\lambda) := [\hat{Z}(\lambda), \hat{H}(\lambda)]$.

Next, inserting the power expansions (16) in this equation, in correspondence to the various perturbative orders, we obtain the following set of conditions:

$$\hat{C}_1 - \mathrm{i}[\hat{Z}_1, \hat{H}_\mathrm{u}] - \hat{H}_\mathrm{p} = 0, \qquad [\hat{C}_1, \hat{H}_\mathrm{u}] = 0$$
$$\hat{C}_2 - \mathrm{i}[\hat{Z}_2, \hat{H}_\mathrm{u}] + \tfrac{1}{2}[\hat{Z}_1, [\hat{Z}_1, \hat{H}_\mathrm{u}]] - \mathrm{i}[\hat{Z}_1, \hat{H}_\mathrm{p}] = 0, \qquad [\hat{C}_2, \hat{H}_\mathrm{u}] = 0$$
$$\vdots$$

where we also have taken into account the additional constraint $[\hat{C}(\lambda), \hat{H}_\mathrm{u}] = 0$. This infinite set of equations can be solved recursively. The first equation, together with the first constraint, determines $\hat{Z}_1$ up to an operator commuting with $\hat{H}_\mathrm{u}$ and $\hat{C}_1$ uniquely and so on. It is convenient to eliminate the arbitrariness in the determination of the operators $\{\hat{Z}_n\}$ choosing the *minimal solution* characterized by the additional condition $\sum_m \hat{P}_m \hat{Z}_n \hat{P}_m = 0, n = 1, 2, \ldots.$

In our particular case, we have the following identifications:

$$\begin{cases} \hat{H}_\mathrm{u} \equiv \hat{H}_0, \\ \lambda \hat{H}_\mathrm{p} \equiv \hat{H}_B + \hat{H}_\updownarrow. \end{cases} \tag{17}$$

Note that the two (infinitely degenerate) eigenspaces of the unperturbed Hamiltonian $\hat{H}_0$ are associated with the eigenprojectors

$$\{\hat{P}_m\}_{m=g,e} = \{\hat{P}_g \equiv 1_B \otimes (\hat{\sigma}_{11} + \hat{\sigma}_{22}),\ \hat{P}_e \equiv 1_B \otimes (\hat{\sigma}_{33})\}, \tag{18}$$

where

$$1_B \equiv \sum_{n_x, n_y, n_z} \left(\left|\psi^x_{n_x}\right\rangle\left\langle\psi^x_{n_x}\right| \right) \otimes \cdots \otimes \left(\left|\psi^z_{n_z}\right\rangle\left\langle\psi^z_{n_z}\right| \right) \tag{19}$$

is the identity in the vibrational Hilbert space. Accordingly, for the operators $\{\hat{C}_1, \hat{Z}_1, \hat{C}_2, \hat{Z}_2, \ldots\}$ forming the minimal solution, we get at the first perturbative order the following expressions:

$$\begin{cases} \lambda \hat{C}_1 = \sum_{m=e,g} \hat{P}_m (\hat{H}_B + \hat{H}_\updownarrow) \hat{P}_m, \\ \lambda \hat{Z}_1 = \mathrm{i} \sum_{j \neq l} \epsilon_{jl} \hat{P}_j (\hat{H}_B + \hat{H}_\updownarrow) \hat{P}_l, \end{cases} \tag{20}$$

with $\epsilon_{ge} = 1 = -\epsilon_{eg}$. Similarly, at the second order, we have

$$\begin{cases} \lambda^2 \hat{C}_2 = \sum_{m=e,g} \hat{P}_m \{\mathrm{i}\lambda[\hat{Z}_1, \hat{H}_B + \hat{H}_\updownarrow] - \frac{1}{2}\lambda^2[\hat{Z}_1, [\hat{Z}_1, \hat{H}_0]]\} \hat{P}_m, \\ \lambda^2 \hat{Z}_2 = \mathrm{i} \sum_{j \neq l} \epsilon_{jl} \hat{P}_j \{\mathrm{i}\lambda[\hat{Z}_1, \hat{H}_B + \hat{H}_\updownarrow] - \frac{1}{2}\lambda^2[\hat{Z}_1, [\hat{Z}_1, \hat{H}_0]]\} \hat{P}_l. \end{cases} \tag{21}$$

Eventually, performing explicit calculations, we find that

$$\lambda \hat{C}_1 = \hat{H}_B, \tag{22}$$

$$\lambda^2 \hat{C}_2 = -\frac{|g_{13}|^2}{\Delta^2}\hat{\sigma}_{11} - \frac{|g_{23}|^2}{\Delta^2}\hat{\sigma}_{22} + \frac{|g_{13}|^2 + |g_{23}|^2}{\Delta^2}\hat{\sigma}_{33} - \left(\frac{g_{13} g_{32}}{\Delta^2} \mathrm{e}^{-\mathrm{i}\vec{k}_{13}\cdot\vec{r}}\, \mathrm{e}^{\mathrm{i}\vec{k}_{23}\cdot\vec{r}} \hat{\sigma}_{12} + \mathrm{h.c.} \right), \tag{23}$$

where we have set $g_{3j} \equiv g^*_{j3}$, and

$$\lambda \hat{Z}_1 = \mathrm{i}\left(\frac{g_{13}}{\Delta} \mathrm{e}^{-\mathrm{i}\vec{k}_{13}\cdot\vec{r}} \hat{\sigma}_{13} - \mathrm{h.c.} \right) + \mathrm{i}\left(\frac{g_{23}}{\Delta} \mathrm{e}^{-\mathrm{i}\vec{k}_{23}\cdot\vec{r}} \hat{\sigma}_{23} - \mathrm{h.c.} \right), \tag{24}$$

$$\lambda^2 \hat{Z}_2 = \frac{\nu}{\Delta} \left\{ \left(\frac{g_{13}}{\Delta} \hat{X}_{13} \hat{\sigma}_{13} + \frac{g_{31}}{\Delta} \hat{X}_{31} \hat{\sigma}_{31} \right) + \left(\frac{g_{23}}{\Delta} \hat{X}_{23} \hat{\sigma}_{23} + \frac{g_{32}}{\Delta} \hat{X}_{32} \hat{\sigma}_{32} \right) \right\}, \tag{25}$$

where

$$\begin{cases} \hat{X}_{j3} := \mathrm{i}\left[\mathrm{e}^{-\mathrm{i}\vec{k}_{j3}\cdot\vec{r}}, \sum_{\alpha=x,y,z} \hat{a}^\dagger_\alpha \hat{a}_\alpha\right], \\ \hat{X}_{3j} := \mathrm{i}\left[\mathrm{e}^{\mathrm{i}\vec{k}_{j3}\cdot\vec{r}}, \sum_{\alpha=x,y,z} \hat{a}^\dagger_\alpha \hat{a}_\alpha\right] = \hat{X}^\dagger_{j3}, \end{cases}$$

with $j = 1, 2$.

The interpretation of this result leads to a very interesting fact. Indeed, it turns out that once the unitary transformation $e^{iZ(\lambda)}$ has been applied to the rotating frame Hamiltonian $\hat{\mathfrak{H}}$ (recall equation (10)), the time evolution of the system is described, at the second order in the parameter λ, by the Hamiltonian

$$\hbar\Delta(\hat{H}_0 + \lambda\hat{C}_1 + \lambda^2\hat{C}_2) = \hat{\mathfrak{H}}_{12} + \hat{\mathfrak{H}}_3, \tag{26}$$

where, in order to display a more transparent formula, we set

$$\hat{\mathfrak{H}}_{12} := \hbar\nu \sum_{\alpha=x,y,z} \left(\hat{a}^\dagger_\alpha \hat{a}_\alpha\right) \otimes (\hat{\sigma}_{11} + \hat{\sigma}_{22}) + \hbar\breve{\omega}_1\hat{\sigma}_{11} + \hbar\breve{\omega}_2\hat{\sigma}_{22} + [\hbar g_{12}\, e^{-i\vec{k}_{12}\cdot\vec{r}}\hat{\sigma}_{12} + \text{h.c.}], \tag{27}$$

$$\hat{\mathfrak{H}}_3 := \hbar\nu \sum_{\alpha=x,y,z} \left(\hat{a}^\dagger_\alpha \hat{a}_\alpha\right) \otimes \hat{\sigma}_{33} + \hbar(\Delta + \breve{\omega}_3)\hat{\sigma}_{33}, \tag{28}$$

with

$$\breve{\omega}_j = -\frac{|g_{j3}|^2}{\Delta}, \quad j = 1, 2, \qquad \breve{\omega}_3 = \frac{|g_{13}|^2 + |g_{23}|^2}{\Delta},$$
$$g_{12} = \frac{g_{13}g_{32}}{\Delta}, \qquad \vec{k}_{12} = \vec{k}_{13} - \vec{k}_{23}.$$

Thus, the transformed Hamiltonian is the sum of two decoupled Hamiltonians $\hat{\mathfrak{H}}_{12}$ and $\hat{\mathfrak{H}}_3$, $[\hat{\mathfrak{H}}_{12}, \hat{\mathfrak{H}}_3] = 0$, 'living', respectively, in the ranges of the orthogonal projectors $\hat{P}_g$ and $\hat{P}_e$. This is a consequence of the fact that $[\hat{P}_m, \hat{C}_n] = 0, m = g, e, n = 1, 2, \ldots$. It is worth noting that the Hamiltonian $\hat{\mathfrak{H}}_{12}$ can be considered as the rotating frame Hamiltonian of a trapped two-level ion in interaction with a laser field characterized by the following parameters:

$$\begin{cases} \omega_{12} \equiv \omega_{13} - \omega_{23} = \omega_2 - \omega_1, \\ \vec{k}_{12} \equiv \vec{k}_{13} - \vec{k}_{23}. \end{cases} \tag{29}$$

This effective coupling can be compared with the result found performing the adiabatic elimination of the level $|3\rangle$ (see [9]). We will come back to this point in the next section.

4. Dynamics of the Raman scheme

The question of what the *complete* dynamics of the system is now arises. First, it will be convenient to adopt the following notation. Given a couple of functions f and h of the perturbative parameter λ, if $f(\lambda) = h(\lambda) + O(\lambda^3)$, we will simply write

$$f(\lambda) \overset{\lambda^2}{\approx} h(\lambda).$$

Next, let us denote by $\hat{T}_\Lambda$ the evolution operator associated with the Raman scheme:

$$i\hbar\left(\frac{d}{dt}\hat{T}_\Lambda\right)(t) = \hat{H}_\Lambda(t)\hat{T}_\Lambda(t), \qquad \hat{T}_\Lambda(0) = \text{Id}. \tag{30}$$

Expressing $\hat{T}_\Lambda$ in terms of the evolution operator associated with the rotating frame Hamiltonian yields

$$\hat{T}_\Lambda(t) = \hat{R}(t)\exp\left(-\frac{i}{\hbar}\hat{\mathfrak{H}}t\right). \tag{31}$$

Now, according to what we have shown in the previous section, we have

$$\begin{aligned} \hat{T}(t) &:= \exp\left(-\frac{i}{\hbar}\hat{\mathfrak{H}}t\right) \\ &= e^{-i\hat{Z}(\lambda)}\exp(-i\Delta\, e^{i\hat{Z}(\lambda)}\hat{H}(\lambda)\, e^{-i\hat{Z}(\lambda)}t)\, e^{i\hat{Z}(\lambda)} \\ &\overset{\lambda^2}{\approx} e^{-i(\lambda\hat{Z}_1+\lambda^2\hat{Z}_2)}\, e^{-i\Delta(\hat{H}_0+\lambda\hat{C}_1+\lambda^2\hat{C}_2)t}\, e^{i(\lambda\hat{Z}_1+\lambda^2\hat{Z}_2)}, \end{aligned} \tag{32}$$

where we have truncated the power expansions of $\hat{Z}(\lambda)$ and $\hat{C}(\lambda)$ at the second order in λ. Formula (32) provides an approximate expression of the evolution operator in the remarkable form of a one-parameter group of unitary transformations. Nevertheless, in order to achieve an approximate expression allowing a direct comparison with the coarse grained dynamics, we still need to perform some manipulation. To this aim, observe that, since the commutators $[\lambda^2\hat{C}_2, \lambda\hat{Z}_1 + \lambda^2\hat{Z}_2]$ and $[\lambda^2\hat{C}_2, \hat{H}_0 + \lambda\hat{C}_1] = [\lambda^2\hat{C}_2, \lambda\hat{C}_1]$ are of the third order in λ, maintaining our degree of approximation we can write

$$\mathrm{e}^{-\mathrm{i}(\lambda\hat{Z}_1+\lambda^2\hat{Z}_2)}\,\mathrm{e}^{-\mathrm{i}\Delta(\hat{H}_0+\lambda\hat{C}_1+\lambda^2\hat{C}_2)t} \overset{\lambda^2}{\approx} \mathrm{e}^{-\mathrm{i}(\lambda\hat{Z}_1+\lambda^2\hat{Z}_2)}\,\mathrm{e}^{-\mathrm{i}\Delta\lambda^2\hat{C}_2 t}\,\mathrm{e}^{-\mathrm{i}\Delta(\hat{H}_0+\lambda\hat{C}_1)t}$$
$$\overset{\lambda^2}{\approx} \mathrm{e}^{-\mathrm{i}\Delta\lambda^2\hat{C}_2 t}\,\mathrm{e}^{-\mathrm{i}(\lambda\hat{Z}_1+\lambda^2\hat{Z}_2)}\,\mathrm{e}^{-\mathrm{i}\Delta(\hat{H}_0+\lambda\hat{C}_1)t}.$$

Therefore, we can manipulate the second-order expression (32) of $\hat{T}(t)$ as follows:

$$\hat{T}(t) \overset{\lambda^2}{\approx} \mathrm{e}^{-\mathrm{i}\Delta\lambda^2\hat{C}_2 t}\,\mathrm{e}^{-\mathrm{i}(\lambda\hat{Z}_1+\lambda^2\hat{Z}_2)}\,\mathrm{e}^{-\mathrm{i}\Delta(\hat{H}_0+\lambda\hat{C}_1)t}\,\mathrm{e}^{\mathrm{i}(\lambda\hat{Z}_1+\lambda^2\hat{Z}_2)}$$
$$= \mathrm{e}^{-\mathrm{i}\Delta\lambda^2\hat{C}_2 t}\,\mathrm{e}^{-\mathrm{i}\Delta(\hat{H}_0+\lambda\hat{C}_1)t}\mathrm{e}^{\mathrm{i}\Delta(\hat{H}_0+\lambda\hat{C}_1)t}\,\mathrm{e}^{-\mathrm{i}(\lambda\hat{Z}_1+\lambda^2\hat{Z}_2)}\,\mathrm{e}^{-\mathrm{i}\Delta(\hat{H}_0+\lambda\hat{C}_1)t}\,\mathrm{e}^{\mathrm{i}(\lambda\hat{Z}_1+\lambda^2\hat{Z}_2)}.$$

Finally, we find a remarkable decomposition of $\hat{T}$:

$$\hat{T}(t) \overset{\lambda^2}{\approx} \hat{T}_\mathrm{e}(t)\hat{T}_\mathrm{f}(t), \tag{33}$$

where we have set

$$\hat{T}_\mathrm{e}(t) := \exp(-\mathrm{i}\Delta(\hat{H}_0 + \lambda\hat{C}_1 + \lambda^2\hat{C}_2)t), \tag{34}$$
$$\hat{T}_\mathrm{f}(t) := \exp(-\mathrm{i}(\lambda\hat{Z}_1(t) + \lambda^2\hat{Z}_2(t)))\exp(\mathrm{i}(\lambda\hat{Z}_1 + \lambda^2\hat{Z}_2)), \tag{35}$$

with $\hat{Z}_k(t) \equiv \mathrm{e}^{\mathrm{i}\Delta(\hat{H}_0+\lambda\hat{C}_1)t}\,\hat{Z}_k\,\mathrm{e}^{-\mathrm{i}\Delta(\hat{H}_0+\lambda\hat{C}_1)t}, k = 1, 2$. It is worth emphasizing that by a completely analogous procedure[4], we also get

$$\hat{T}(t) \overset{\lambda^2}{\approx} \hat{T}'_\mathrm{f}(t)\hat{T}_\mathrm{e}(t), \qquad \hat{T}'_\mathrm{f}(t) := \hat{T}_\mathrm{f}(-t)^\dagger. \tag{36}$$

Note that, due to the specific dependence of $\hat{T}_\mathrm{f}$ on t, one has $\hat{T}'_\mathrm{f}(t) \neq \hat{T}_\mathrm{f}(t)$.

In the light of formula (33) or (36), the rotating frame time evolution given by $\hat{T}$ may be thought of as a process consisting of two fundamental components. One of these is an effective time evolution, described by $\hat{T}_\mathrm{e}$, in which levels $|1\rangle$ and $|2\rangle$ are decoupled from level $|3\rangle$. The other component, the one $\hat{T}_\mathrm{f}$ (or $\hat{T}'_\mathrm{f}$) is responsible for, is a correction to $\hat{T}_\mathrm{e}$ and involves fast transitions (the operators $\hat{Z}_1(t)$ and $\hat{Z}_2(t)$ oscillate at the detuning frequency Δ) from and to the third atomic level. Considering the complete time evolution (31), observe that the unitary evolution described by $\hat{R}\hat{T}_\mathrm{e}$ corresponds to the effective dynamics obtained in [9] restricting the analysis to the coarse grained observables. In fact, $\hat{R}\hat{T}_\mathrm{e}$ is the evolution operator associated with the time-dependent effective Hamiltonian

$$\hat{H}_\mathrm{e} = \hat{H}_\mathrm{e}^{(12)} + \hat{H}_\mathrm{e}^{(3)}, \tag{37}$$

where

$$\hat{H}_\mathrm{e}^{(12)}(t) := \hbar\nu \sum_{\alpha=x,y,z} \left(\hat{a}_\alpha^\dagger\hat{a}_\alpha\right) \otimes (\hat{\sigma}_{11} + \hat{\sigma}_{22}) + \hbar(\omega_1 + \breve{\omega}_1)\hat{\sigma}_{11} + \hbar(\omega_2 + \breve{\omega}_2)\hat{\sigma}_{22}$$
$$+ [\hbar g_{12}\,\mathrm{e}^{-\mathrm{i}(\vec{k}_{12}\cdot\vec{r}-\omega_{12}t)}\hat{\sigma}_{12} + \mathrm{h.c.}], \tag{38}$$
$$\hat{H}_\mathrm{e}^{(3)} := \hbar\nu \sum_{\alpha=x,y,z} \left(\hat{a}_\alpha^\dagger\hat{a}_\alpha\right) \otimes \hat{\sigma}_{33} + \hbar(\omega_3 + \breve{\omega}_3)\hat{\sigma}_{33}. \tag{39}$$

[4] The calculation may be carried out *directly*, i.e., step by step as in the previous case but changing the reordering of the exponentials in (32) and subsequent formulae, or from (33) exploiting the fact that $\hat{T}(t) = \hat{T}(-t)^\dagger$.

We remark that we have deduced this result analytically; no adiabatic approximation has been performed. We also stress that a relevant difference between $\hat{T}_e$ and $\hat{T}_f$ consists in the kind of time dependence. Indeed, on one hand, the unitary evolution $\hat{T}_e$ forms a one-parameter group, hence it is expressible as the exponential of a generator multiplied by t. It follows that a truncated power expansion of the exponential retains its validity only on a finite time span. On the other hand, $\hat{T}_f$ can be expressed as the exponential of an operator whose time dependence involves only sinusoidal factors. In fact, we have

$$\hat{T}_f(t) \overset{\lambda^2}{\approx} e^{-i(\lambda(\hat{Z}_1(t)-\hat{Z}_1)+\lambda^2(\hat{Z}_2(t)-\hat{Z}_2))}\, e^{\frac{1}{2}[\lambda\hat{Z}_1(t)+\lambda^2\hat{Z}_2(t),\lambda\hat{Z}_1+\lambda^2\hat{Z}_2]}$$
$$\overset{\lambda^2}{\approx} e^{-i(\lambda(\hat{Z}_1(t)-\hat{Z}_1)+\lambda^2(\hat{Z}_2(t)-\hat{Z}_2))}\, e^{\frac{1}{2}\lambda^2[\hat{Z}_1(t),\hat{Z}_1]}.$$

It then follows that the truncated expansion

$$\begin{aligned}\hat{T}_f(t) &\overset{\lambda^2}{\approx} 1 - i\lambda(\hat{Z}_1(t) - \hat{Z}_1) - \lambda^2\big(i(\hat{Z}_2(t) - \hat{Z}_2) + \tfrac{1}{2}(\hat{Z}_1(t) - \hat{Z}_1)^2 - \tfrac{1}{2}[\hat{Z}_1(t), \hat{Z}_1]\big)\\ &= 1 - i\lambda(\hat{Z}_1(t) - \hat{Z}_1) - \lambda^2\big(\tfrac{1}{2}\big(\hat{Z}_1(t)^2 + \hat{Z}_1^2\big) - \hat{Z}_1(t)\hat{Z}_1 + i(\hat{Z}_2(t) - \hat{Z}_2)\big)\end{aligned} \tag{40}$$

is legitimated independently on time.

Let us summarize the results obtained up to now. We have shown that the standard adiabatic elimination technique provides an effective dynamics, described by $\hat{R}\hat{T}_e$, that differs from the complete second-order dynamics of the Raman scheme for the presence of another unitary evolution which can be cast in the form of the exponential of a rapidly oscillating operator function of time. Therefore, the factorization into a coarse grained and a fine dynamics given by equation (33) makes the correction to the adiabatic approximation solution very readable and easy to be calculated, in view of the truncated expression (40). It is worth noting that the corrections due to $\hat{T}_f$ are associated with the operators $\lambda\hat{Z}_1, \lambda\hat{Z}_1(t)$ and $\lambda^2\hat{Z}_2, \lambda^2\hat{Z}_2(t)$, which are, respectively, of the first- and second-order in the perturbative parameter; thus, they cannot be neglected at a fine time scale. Precisely, they provide terms oscillating at the detuning frequency which couple levels $|1\rangle$ and $|2\rangle$ with the auxiliary level. Hence, as expected, the fine dynamics is very fast. Such a detailed knowledge of the complete dynamics should become of practical interest in connection with time resolution improvements of experiments. Moreover, if the auxiliary level $|3\rangle$ is an excited level with non-negligible decay rates towards the other two levels, one should expect that these fast transitions, in association with decays, can become an important source of decoherence for the system, with observable effects at a time scale much larger than $|\Delta|^{-1}$.

5. Decoherence effects

Up to this point, we have considered the coherent dynamics only of the trapped ion Raman scheme. However, as anticipated in section 2, there is experimental evidence that the decoherence effects can play a non-negligible role. Precisely, in observations of Rabi oscillations of a trapped $^9Be^+$, significant damping effects, increasing with increasing motional excitation, have been reported [12]. It has been suggested that the main decoherence source could be technical noise, but the analysis of various possible mechanisms shows that they are too small to explain the observations [16]. Di Fidio and Vogel [17] have then proposed (and we share their idea) that the decoherence source is inherent in the physical system itself. By means of the adiabatic elimination of the auxiliary level, they derive a master equation which involves levels $|1\rangle$ and $|2\rangle$, and perform numerical simulations in good agreement with experimental data, except for the dependence of damping on motional excitation.

In our opinion, a more transparent interpretation of these decoherence effects can be given taking into account the fast transitions—let us call them *anomalous transitions*—that couple levels $|1\rangle$ and $|2\rangle$ with the auxiliary level $|3\rangle$ which in current literature (by virtue of the adiabatic elimination) is considered merely as a 'dumb' level. Indeed, in general, the auxiliary level $|3\rangle$ can be an excited level characterized by some decay rates $\gamma_{3\downarrow1}, \gamma_{3\downarrow2} \geqslant 0$ towards levels $|1\rangle$ and $|2\rangle$. Thus, it is quite natural to expect that the compound effect of anomalous transitions and decays from the auxiliary level is to gradually introduce decoherence into the effective coherent cycle involving levels $|1\rangle$ and $|2\rangle$. We will show that this intuition is correct.

In order to take into account the mentioned decoherence effects, let us consider the master equation governing the evolution of the density matrix of the system:

$$\dot{\varrho} = \mathbb{L}(\hbar^{-1}\hat{H}_\Lambda(t); \sqrt{\gamma_{3\downarrow1}}\,\hat{\sigma}_{31}, \sqrt{\gamma_{3\downarrow2}}\,\hat{\sigma}_{32})\varrho, \tag{41}$$

where $\mathbb{L}(\cdot)[\cdot]$ is the *Lindblad superoperator* defined by

$$\mathbb{L}(\hat{G}; \hat{F}_1, \hat{F}_2)\varrho = -\mathrm{i}[\hat{G}, \varrho] + \sum_{j=1,2}\left(\hat{F}_j^\dagger\varrho\hat{F}_j - \frac{1}{2}\{\hat{F}_j\hat{F}_j^\dagger, \varrho\}\right), \tag{42}$$

with $\{\cdot,\cdot\}$ denoting the anti-commutator. Note that, for $\gamma_{3\downarrow1} = \gamma_{3\downarrow2} = 0$, equation (41) reduces to the standard Schrödinger equation of the Raman scheme, while, if at least one of the decay rates is nonzero, it describes the Raman scheme with relaxation of the auxiliary level $|3\rangle$.

As in section 2, the first step is to pass to the interaction picture associated with the canonical transformation $\hat{R}(t)$:

$$\varrho_{\mathrm{int}}(t) = \hat{R}(t)^\dagger\varrho(t)\hat{R}(t). \tag{43}$$

In this picture, since

$$\mathbb{L}(\hat{R}(t)^\dagger(H_\Lambda(t) - \hat{A})\hat{R}(t); \sqrt{\gamma_{3\downarrow1}}\,\hat{R}(t)^\dagger\hat{\sigma}_{31}\hat{R}(t), 1\to2) = \mathbb{L}(\Delta\hat{H}; \sqrt{\gamma_{3\downarrow1}}\,\hat{\sigma}_{31}, 1\to2),$$

the master equation reads

$$\dot{\varrho}_{\mathrm{int}}(t) = \mathbb{L}(\Delta\hat{H}; \sqrt{\gamma_{3\downarrow1}}\,\hat{\sigma}_{31}, \sqrt{\gamma_{3\downarrow2}}\,\hat{\sigma}_{32})\varrho_{\mathrm{int}}(t). \tag{44}$$

At this point, we can use the unitary transformation $\hat{W} \equiv \exp(\mathrm{i}\lambda\hat{Z}_1 + \mathrm{i}\lambda^2\hat{Z}_2)$ described in section 3. Introducing the new effective density operator

$$\varrho_{\mathrm{e}}(t) := \hat{W}\varrho_{\mathrm{int}}(t)\hat{W}^\dagger, \tag{45}$$

we have

$$\dot{\varrho}_{\mathrm{e}}(t) = \mathbb{L}(\hbar^{-1}(\hat{\mathfrak{H}}_{12} + \hat{\mathfrak{H}}_3); \sqrt{\gamma_{3\downarrow1}}\,\hat{W}\hat{\sigma}_{31}\hat{W}^\dagger, 1\to2)\varrho_{\mathrm{e}}(t), \tag{46}$$

$\hat{\mathfrak{H}}_{12}$ and $\hat{\mathfrak{H}}_3$ are defined, respectively, by equations (27) and (28). Now, if we assume that $|\Delta| \gg \gamma_{3\downarrow1}, \gamma_{3\downarrow2}$,[5] it is a good approximation to retain in the power expansion of $\sqrt{\gamma_{3\downarrow1}}\,\hat{W}\hat{\sigma}_{31}\hat{W}^\dagger$ only those terms that are at most linear in the perturbative parameter λ, namely

$$\sqrt{\gamma_{3\downarrow1}}\,\hat{W}\hat{\sigma}_{31}\hat{W}^\dagger \approx \sqrt{\gamma_{3\downarrow1}}\,(\hat{\sigma}_{31} - \kappa_{13}(\hat{\sigma}_{11} - \hat{\sigma}_{33}) - \kappa_{23}\hat{\sigma}_{21}), \tag{47}$$

where

$$\kappa_{j3} \equiv \frac{g_{j3}}{\Delta}\exp(-\mathrm{i}\vec{k}_{j3}\cdot\vec{r}), \qquad j = 1, 2. \tag{48}$$

Obviously, an analogous argument holds for $\sqrt{\gamma_{3\downarrow2}}\,\hat{W}\hat{\sigma}_{32}\hat{W}^\dagger$.

Eventually, for the dynamics of the effective density matrix, we obtain the following master equation:

$$\dot{\varrho}_{\mathrm{e}}(t) = \mathbb{L}(\hbar^{-1}(\hat{\mathfrak{H}}_{12} + \hat{\mathfrak{H}}_3); (c_{31}\hat{\sigma}_{31} + c_{3(1)3}(\vec{r})\hat{\sigma}_{33}) - (c_{21}(\vec{r})\hat{\sigma}_{21} + c_{11}(\vec{r})\hat{\sigma}_{11}), 1 \leftrightarrow 2)\varrho_{\mathrm{e}}(t), \tag{49}$$

where we have set

[5] This condition is actually satisfied in experiments, see [17].

$$c_{31} \equiv \sqrt{\gamma_{3\downarrow 1}}, \qquad c_{3(1)3}(\vec{r}) \equiv \sqrt{\gamma_{3\downarrow 1}}\kappa_{13}, \tag{50}$$

$$c_{21}(\vec{r}) \equiv \kappa_{23}\sqrt{\gamma_{3\downarrow 1}}, \qquad c_{11}(\vec{r}) \equiv \kappa_{13}\sqrt{\gamma_{3\downarrow 1}}, \tag{51}$$

and similar definitions hold with the exchange $1 \leftrightarrow 2$. Note that, due to the dependence on $\vec{r} \propto \left(\hat{a}_x + \hat{a}_x^\dagger, \hat{a}_y + \hat{a}_y^\dagger, \hat{a}_z + \hat{a}_z^\dagger\right)$ of the coefficients (50) and (51), one expects a dependence on motional excitation of the decoherence effects, in agreement with experimental observations [12], as already mentioned.

In our opinion, the result obtained is very expressive. We started from a master equation for the density matrix of the system, equation (41), in which the incoherent component of the dynamics involves decays from the auxiliary level $|3\rangle$ to levels $|1\rangle$ and $|2\rangle$. Therefore, if the anomalous transitions which couple levels $|1\rangle$ and $|2\rangle$ with the auxiliary level played no relevant role in connection with the relaxation of this level, one could expect that the system would evolve from any initial condition towards a coherent dynamics involving levels $|1\rangle$ and $|2\rangle$ only. Experimental observations indicate that this is not the case. In fact, by virtue of the unitary transformation $\hat{W}$, we are able to observe the system in a 'reference frame' in which the coherent dynamics of the subsystem $\{|1\rangle, |2\rangle\}$ is decoupled from that of the auxiliary level, *but* the incoherent component of the dynamics involves population fluxes connecting each couple of levels, with the effect of gradually introducing decoherence in the coherent cycle involving levels $|1\rangle$ and $|2\rangle$. This is the physical content of equation (49). A detailed study of this equation is beyond the scope of this paper. Our purpose for later work is to investigate its solutions by quantum trajectory methods [18].

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